Problem Sheet 3, Part 1

Properties of convolution.

1. Show that for all arithmetic functions f, g and h we have
$$f * (g + h) = f * g + f * h.$$
So $*$ is **distributive** over $+$.

2. Recall $\delta(n) = 1$ if $n = 1$, 0 otherwise. Show that
$$f * \delta = \delta * f = f.$$
for all arithmetic functions f. Thus δ is the **identity** for $*$.

Connections between sq, Q_2, μ_2 and λ

3. Prove two results stated in lectures,

   i) for all $k \geq 2$, $Q_k = 1 * \mu_k$,

   ii) $2^\omega = 1 * Q_2$.

4. Define the arithmetic function sq by
$$sq(n) = \begin{cases} 1 & \text{if } n = m^2 \text{ for some integer } m, \text{ (i.e. } n \text{ is a square)}, \\ 0 & \text{otherwise.} \end{cases}$$
So sq is the characteristic function of the square numbers.

   i) Prove that
$$D_{sq}(s) = \zeta(2s)$$
for $\operatorname{Re} s > 1/2$.

   ii) Recall from Example 3.29 in the notes that for Q_2, the characteristic function of square-free number,
$$\sum_{n=1}^{\infty} \frac{Q_2(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)}.$$

Rearrange this equality and use Part i as

$$\zeta(s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{Q_2(n)}{n^s} = \sum_{n=1}^{\infty} \frac{sq(n)}{n^s} \sum_{n=1}^{\infty} \frac{Q_2(n)}{n^s} = \sum_{n=1}^{\infty} \frac{sq * Q_2(n)}{n^s}$$

by the composition of Dirichlet Series.

a) Why does this suggests that $1 = sq * Q_2$?

b) Prove this directly.

5. Show, by looking at Euler Product for the Left Hand side that,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

for $\mathrm{Re}\, s > 1$, where λ is Liouville's function defined by $\lambda(n) = (-1)^{\Omega(n)}$ for all $n \geq 1$.

6. From the Question 5

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} &= \zeta(2s) \frac{1}{\zeta(s)} \\ &= \sum_{n=1}^{\infty} \frac{sq(n)}{n^s} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \\ &\qquad \text{by Question 4 and definition of } \mu, \\ &= \sum_{n=1}^{\infty} \frac{sq * \mu(n)}{n^s}, \end{aligned}$$

by the convolution of Dirichlet Series. This 'suggests' that $\lambda = sq * \mu$. Prove this directly.

7. In the notes (or special case of Question 3) we have $Q_2 = 1 * \mu_2$. From Question 6 $\lambda = sq * \mu$ and, from Question 4ii, $1 = sq * Q_2$. Use these three results along with Möbius Inversion to prove

   i) $\delta = Q_2 * \lambda$,

ii) $\delta = \mu_2 * sq$.

(Do **not** use the method of showing equality at prime powers.)

Since δ is the identity with respect to $*$ these results show that λ **is the inverse of** Q_2 and sq **the inverse of** μ_2.

iii) What is $1 * \lambda$?

iv) What is $\mu_2 * \lambda$?

Problem Sheet 3, Part 2

Summing series

When studying the individual factors in the Euler Product of the Dirichlet Series associated with an arithmetic function it is often necessary to sum series.

8. Sum the following series, expressing your answers as positive and negative powers of terms of the form $1 - y^m$ for various $m \geq 1$.

   You need not worry about convergence.

   i)

$$\begin{aligned} S_1 &= 1 + y + y^2 + y^3 + y^4 + ... \\ S_{-1} &= 1 - y + y^2 - y^3 + y^4 - ... \end{aligned}$$

   Hint The map from $S_a \to S_{-a}$ is $y \to -y$.

   ii)

$$\begin{aligned} S_2 &= 1 + 2y + 2y^2 + 2y^3 + 2y^4 + ... \\ S_{-2} &= 1 - 2y + 2y^2 - 2y^3 + 2y^4 - ... \end{aligned}$$

   Hint Write S_2 in terms of S_1.

   iii)

$$\begin{aligned} S_3 &= 1 + 2y + 3y^2 + 4y^3 + 5y^4 + ... \\ S_{-3} &= 1 - 2y + 3y^2 - 4y^3 + 5y^4 - ... \end{aligned}$$

   Hint Write $S_3 - S_1$ in terms of S_3.

   iv)

$$\begin{aligned} S_4 &= 1 + 3y + 5y^2 + 7y^3 + 9y^4 + ... \\ S_{-4} &= 1 - 3y + 5y^2 - 7y^3 + 9y^4 - ... \end{aligned}$$

   Hint Write $S_4 - S_3$ in terms of S_3.

v) Prove that

$$S_5 = 1 + 2^2 y + 3^2 y^2 + 4^2 y^3 + 5^2 y^4 + ... = \frac{1-y^2}{(1-y)^4}.$$

Evaluate

$$S_{-5} = 1 - 2^2 y + 3^2 y^2 - 4^2 y^3 + 5^2 y^4 - ...$$

Hint Consider $S_5 - S_4$.

vi)

$$\begin{aligned} S_6 &= 1 + y^2 + y^3 + y^4 + y^5 + ... \\ S_{-6} &= 1 + y^2 - y^3 + y^4 - y^5 + ... \end{aligned}$$

9. Factorise the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\lambda(n)\, 2^{\omega(n)}}{n^s},$$

where λ is Liouville's function $\lambda(n) = (-1)^{\Omega(n)}$.

10. The results $2^\omega = 1 * Q_2$ and $Q_2 = 1 * \mu_2$ combine to give

$$2^\omega = d * \mu_2.$$

i) Prove this by showing that we have equality on any power of a prime.

ii) Deduce that

a) $d = sq * 2^\omega$.

b) $1 = \lambda * 2^\omega$.

Do **not** use the method of showing equality on prime powers, but rather use all previous results along with Möbius Inversion.

11. i) Factor the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}.$$

ii) Factor the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\lambda(n)\, d(n^2)}{n^s}.$$

12. In lectures we define the arithmetic function g by $g(n) = d(n^2)$ for all $n \geq 1$. (*This is non-standard notation*) So the previous question can be written as $g = 1 * 2^\omega$.

 Using this result from the last question prove that

 i) $g = d * Q_2$,

 ii) $g = d_3 * \mu_2$,

 iii) $d = \lambda * g$,

 iv) $d_3 = sq * g$.

 (Do **not** use the method of showing equality at prime powers, but use all previous results along with Möbius Inversion.)

13. Factor the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\lambda(n)\, d^2(n)}{n^s}$$

14. Using $d^2 = 1 * g$ prove that

 i) $d^2 = d * 2^\omega$,

 ii) $d^2 = d_3 * Q_2$,

 iii) $d^2 = d_4 * \mu_2$,

 iv) $d_4 = sq * d^2$.

 v) $d_3 = \lambda * d^2$,

 (Do **not** use the method of showing equality at prime powers, but use all previous results along with Möbius Inversion.)

Problem Sheet 3, Part 3

The arithmetic function $q_2, \lambda q_2, \sigma$ and ϕ

15. Define a **square-full** number to be one for which if $p|n$ then $p^2|n$. In other words n is square-full iff $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r}$ with distinct primes and $a_i \geq 2$ for all $1 \leq i \leq r$. Let q_2 be the characteristic function for square-full numbers, so $q_2(n) = 1$ if n is square-full, 0 otherwise.

    i) Prove that
    $$\sum_{n=1}^{\infty} \frac{q_2(n)}{n^s} = \frac{\zeta(2s)\,\zeta(3s)}{\zeta(6s)}$$
    for $\mathrm{Re}\, s > 1/2$.

    Hint Look at the Euler Product of the Dirichlet Series on the left hand side.

    ii) Find a similar expression for
    $$\sum_{n=1}^{\infty} \frac{\lambda(n)\, q_2(n)}{n^s}$$

16. In the notes we have, by definition, $\sigma = 1 * j$ while we showed that $\phi = \mu * j$ followed from the definition of ϕ. Use these two decompositions to prove

    i)
    $$\text{a)} \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\,\zeta(s-1) \quad \text{and} \quad \text{b)} \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$
    for $\mathrm{Re}\, s > 2$.

    ii)
    $$\text{a)}\ \phi * \sigma = j * j \qquad \text{and} \qquad \text{b)}\ d * \phi = \sigma.$$

    Hint do **not** check equality of prime powers.

17. *Results from this question are used in the next Problem Sheet.*

Prove, by showing that both sides of each expression are equal on prime powers that

$$\text{i) } \frac{\phi}{j} = 1 * \frac{\mu}{j}, \qquad \text{ii) } \frac{\sigma}{j} = 1 * \frac{1}{j} \qquad \text{and} \qquad \text{iii) } \frac{j}{\phi} = 1 * \frac{Q_2}{\phi}.$$

Hint If you can't recall it, look back in the notes, but it suffices to show that $f(p^a) - f(p^{a-1}) = g(p^a)$ for all $a \geq 1$.

Note that if $a = 1$ then, since f is multiplicative, the condition that needs to be checked is that $f(p) - 1 = g(p)$.

Problem Sheet 3, Part 4

A few final results.

18. Show, by looking at the Euler Products for the Left Hand Sides, that

$$\text{i) } \frac{\zeta(2)}{\zeta(3)} = \prod_p \left(1 + \frac{1}{p\,(p+1)}\right), \qquad \text{ii) } \frac{\zeta(2)}{\zeta(4)} = \prod_p \left(1 + \frac{1}{p^2}\right),$$

$$\text{iii) } \frac{\zeta(3)}{\zeta(6)} = \prod_p \left(1 + \frac{1}{p^3}\right), \qquad \text{iv) } \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} = \prod_p \left(1 + \frac{1}{p\,(p-1)}\right).$$

19. *Results from this question are used in the next Problem Sheet.*

Using Euler Products show that

$$\frac{6}{\pi^2} = \prod_p \left(1 - \frac{1}{p^2}\right) \le \frac{\phi(n)\,\sigma(n)}{n^2} \le 1$$

for all $n \ge 1$.

You may assume that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

20. Prove that

$$\sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ \gcd(m,n)=1}}^{\infty} \frac{1}{(mn)^2} = \frac{\zeta^2(2)}{\zeta(4)}.$$

In fact, it can be shown that the right hand side equals 5/2.

Hint Use the arithmetic function δ to pick out the condition $\gcd(m,n) = 1$. Then use Möbius Inversion on δ. This method was used in the notes to prove that $\phi(n) = \sum_{d|n} \mu(d)\, n/d$.